

Re-Scaling of Energy in the Stringy Charged Black Hole Solutions using Approximate Symmetries

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Abstract

This paper is devoted to study the energy problem in general relativity using approximate Lie symmetry methods for differential equations. We evaluate second-order approximate symmetries of the geodesic equations for the stringy charged black hole solutions. It is concluded that energy must be re-scaled by some factor in the second-order approximation.

Keywords: Stringy charged black holes; Approximate symmetries; Energy re-scaling.

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1 Introduction

One of the most important issues of General Relativity (GR) is the localization of energy-momentum. It is a conserved quantity in classical mechanics. In GR, energy conservation is guaranteed only for those spacetimes which are time translational invariant. In that case, energy is defined as the dot product of timelike Killing vector \mathbf{k} and momentum 4-vector \mathbf{p} , i.e., $E = \mathbf{k} \cdot \mathbf{p}$.

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However, there are numerous spacetimes which are non-stationary (admitting no timelike isometry and consequently energy is not conserved there) [1]. For the case of gravitational waves, this problem is of special interest as $T_{\mu\nu} = 0$ and hence energy is not given by the energy-momentum tensor. The existence of gravitational waves was demonstrated by Weber and Wheeler [2] and Ehlers and Kundt [3]. They proposed an approximate formula to check the momentum imparted by gravitational waves to test particles placed in their path. Using the pseudo-Newtonian formalism, an exact formula was presented by Qadir and Sharif [4]. They showed that the results obtained for plane and cylindrical gravitational waves coincide with those found by the above people.

In order to have a trim expression for energy and momentum in GR, Einstein [5] himself proposed a prescription which is a combination of the energy-momentum tensor and the pseudo-tensor. Many others including Landau-Lifshitz, Papapetrou, Weinberg etc. [6]-[8] also developed energy-momentum complexes. All these prescriptions are coordinate dependent providing meaningful results only in Cartesian coordinates. In order to overcome this difficulty, Møller, Bondi, Ashtekar-Hansen [9]-[11] and others formulated coordinate independent prescriptions. It has been shown that for a given spacetime, various prescriptions not necessarily provide the same result. However, none of these attempts proved to be unambiguous and consequently could not yield a correct definition. Therefore it would be interesting to develop some procedure which should be more accurate than the previous ones.

The spacetime symmetries are characterized by its isometries which form a Lie algebra [12]. The symmetries of differential equations (DEs) also form Lie algebra [13]. Since the geodesic equations are the DEs therefore it has been suggested that there would be a nexus between spacetime isometries and the symmetries of the geodesic equations [14, 15]. The symmetries of the manifold which are obtained through the geodesic equations yield some additional symmetries which provide no conservation law. However, other symmetries yield quantities conserved under geodesic motion as well as first integrals of the geodesic equations [16, 17]. A manifold with no exact symmetry can possess approximate symmetries and more interesting information can be obtained from a slightly broken (approximate) symmetry than the exact symmetry. There are different methods available to find approximate symmetries of DEs. Two “approximate symmetry” theories have been constructed by using the combination of Lie group theory and perturbations. The first theory was proposed by Baikov et al. [18] and the second theory

was given by Fushchich and Shtelen [19].

Kara et al. [20] used the first approximate symmetry method to discuss conservation laws of energy and momentum for the Schwarzschild spacetime. Later, this procedure was applied to the Reissner-Nordström (RN) solution [21] and an energy re-scaling factor was obtained in the second-order approximation. Hussain et al. [22, 23] found energy re-scaling factors for the Kerr-Newmann and Kerr-Newmann AdS spacetimes by using the same procedure.

In going from the Minkowski spacetime (flat) to non-flat stringy charged solutions, the recovered approximate symmetries and consequently the conservation laws (recovered in the first-order approximation) are lost. It is expected that in the limit of small charge we should retrieve all the lost symmetries. For this purpose, we have to find out the second order approximate symmetries. In this paper, we use the first procedure of approximate symmetry to evaluate approximate symmetries and to obtain energy re-scaling factor for the stringy charged black holes.

The paper is organized as follows. In the next section, we discuss exact and approximate symmetry methods for the solution of DEs. Section **3** contains exact symmetries of the Minkowski spacetime and the first-order approximate symmetries of the Schwarzschild spacetime. In section **4**, we study approximate symmetries of the stringy charged black hole solutions, i.e., electrically and magnetically charged solutions. Finally, we summarize and discuss the results in the last section.

2 Mathematical Formulation

Here we review approximate symmetry analysis for the solution of DEs. Symmetry is a point transformation, it may be independent of the choice of variables, which maps the solutions into the solutions. In other words, one can say that symmetries are the transformations under which the form of DEs does not change. A point transformation is a transformation which maps one point (x, y) into another point (x^*, y^*) . Symmetries are very useful in finding the solution of DEs or reducing them to the more simpler form for integration. For example, with the help of symmetries, one can convert non-linear DEs which arise in many practical problems into linear DEs. Also, the importance of symmetries lies in the most celebrated Noether's theorem which is stated as "Any differentiable symmetry of the action of a physical

system has a corresponding conservation law" [24, 25]. If, for a given system of DEs, there is a variational principle, then a continuous symmetry invariant under the action of the functional provides a conservation law [26]-[28].

The symmetry generator of a n^{th} -order ordinary differential equation (ODE) (involving s as independent variable and \mathbf{x} as dependent variable and \mathbf{x}' , \mathbf{x}'' , ..., $\mathbf{x}^{(n)}$) represents its first, second and so on n^{th} -order derivative with respect to s) [13, 29]

$$\mathbf{E}(s; \mathbf{x}(s), \mathbf{x}'(s), \mathbf{x}''(s), \dots, \mathbf{x}^{(n)}(s)) = 0 \quad (1)$$

under the point transformation: $(s, \mathbf{x}) \longrightarrow (\xi(s, \mathbf{x}), \eta(s, \mathbf{x}))$ can be found if on the solution of the ODE, $\mathbf{E} = 0$, the following condition is satisfied

$$\mathbf{X}^{[n]}(\mathbf{E})|_{\mathbf{E}=0} = 0.$$

Here, $\mathbf{X}^{[n]}$ is the n^{th} -order prolongation or extension of the infinitesimal symmetry generator

$$\mathbf{X} = \xi(s, \mathbf{x}) \frac{\partial}{\partial s} + \eta(s, \mathbf{x}) \frac{\partial}{\partial \mathbf{x}} \quad (2)$$

and is given by

$$\begin{aligned} \mathbf{X}^{[n]} &= \xi(s, \mathbf{x}) \frac{\partial}{\partial s} + \eta(s, \mathbf{x}) \frac{\partial}{\partial \mathbf{x}} + \eta_{,s}(s, \mathbf{x}, \mathbf{x}') \frac{\partial}{\partial \mathbf{x}'} + \dots \\ &+ \eta_{,(n)}(s, \mathbf{x}, \mathbf{x}', \dots, \mathbf{x}^{(n)}) \frac{\partial}{\partial \mathbf{x}^{(n)}}. \end{aligned} \quad (3)$$

The values of the prolongation coefficients are given by

$$\eta_{,s} = \frac{d\eta}{ds} - \mathbf{x}' \frac{d\xi}{ds}, \quad \eta_{,(n)} = \frac{d\eta_{,(n-1)}}{ds} - \mathbf{x}^{(n)} \frac{d\xi}{ds}; \quad n \geq 2. \quad (4)$$

In the same manner, the system of p ODEs of order n

$$\mathbf{E}_\alpha(s; \mathbf{x}(s), \mathbf{x}'(s), \mathbf{x}''(s), \dots, \mathbf{x}^{(n)}(s)) = 0, \quad (\alpha = 1, 2, 3, \dots, p) \quad (5)$$

admits a symmetry generator

$$\mathbf{X} = \xi(s, \mathbf{x}) \frac{\partial}{\partial s} + \eta^\alpha(s, \mathbf{x}) \frac{\partial}{\partial \mathbf{x}^\alpha} \quad (6)$$

if and only if for each ODE, the following symmetry condition

$$\mathbf{X}^{[n]}(\mathbf{E}_\alpha)|_{\mathbf{E}_\alpha=0} = 0$$

is satisfied. In that case, n^{th} -order extension of the symmetry generator (6) is given by

$$\begin{aligned} \mathbf{X}^{[n]} &= \xi(s, \mathbf{x}) \frac{\partial}{\partial s} + \eta^\alpha(s, \mathbf{x}) \frac{\partial}{\partial \mathbf{x}^\alpha} + \eta_{,s}^\alpha(s, \mathbf{x}, \mathbf{x}') \frac{\partial}{\partial \mathbf{x}^{\alpha'}} + \dots \\ &+ \eta_{,(n)}^\alpha(s, \mathbf{x}, \mathbf{x}', \dots, \mathbf{x}^{(n)}) \frac{\partial}{\partial \mathbf{x}^{\alpha(n)}} \end{aligned} \quad (7)$$

and the corresponding prolongation coefficients are

$$\eta_{,s}^\alpha = \frac{d\eta^\alpha}{ds} - \mathbf{x}^{\alpha'} \frac{d\xi}{ds}, \quad \eta_{,(n)}^\alpha = \frac{d\eta_{,(n-1)}^\alpha}{ds} - \mathbf{x}^{\alpha(n)} \frac{d\xi}{ds}; \quad n \geq 2. \quad (8)$$

If $p = 1$, then the system reduces to a single equation. The k^{th} -order approximate symmetry of a perturbed system of ODEs

$$\mathbf{E} = \mathbf{E}_0 + \epsilon \mathbf{E}_1 + \epsilon^2 \mathbf{E}_2 + \dots + \epsilon^k \mathbf{E}_k + O(\epsilon^{k+1}) \quad (9)$$

is given by the generator

$$\mathbf{X} = \mathbf{X}_0 + \epsilon \mathbf{X}_1 + \epsilon^2 \mathbf{X}_2 + \dots + \epsilon^k \mathbf{X}_k \quad (10)$$

if the following symmetry condition holds [30]

$$\begin{aligned} \mathbf{X}\mathbf{E} &= [(\mathbf{X} = \mathbf{X}_0 + \epsilon \mathbf{X}_1 + \epsilon^2 \mathbf{X}_2 + \dots + \epsilon^k \mathbf{X}_k)(\mathbf{E} = \mathbf{E}_0 + \epsilon \mathbf{E}_1 \\ &+ \epsilon^2 \mathbf{E}_2 + \dots + \epsilon^k \mathbf{E}_k)] \big|_{\mathbf{E}=\mathbf{E}_0+\epsilon\mathbf{E}_1+\epsilon^2\mathbf{E}_2+\dots+\epsilon^k\mathbf{E}_k} = O(\epsilon^{k+1}). \end{aligned} \quad (11)$$

Here $\epsilon \in R^+$, \mathbf{E}_0 is the exact system of equations, \mathbf{E}_1 , \mathbf{E}_2 are the first and second order perturbed parts of the perturbed DEs respectively and so on. \mathbf{X}_0 represents the exact part of the symmetry generator and \mathbf{X}_1 , \mathbf{X}_2 denote respectively the first and second order approximate parts of the symmetry generator and so on. For the k^{th} -order approximate symmetry generator, we put terms involving $\epsilon^{(k+1)}$ and its higher powers equal to zero (then the RHS of Eq.(11) becomes zero). We know that symmetries of an ODE always form a Lie algebra. However, the approximate symmetries do not necessarily form a Lie algebra rather do form the so-called "approximate Lie algebra" (up to a specified degree of precision) [31]. The perturbed equation always admits an approximate symmetry $\epsilon \mathbf{X}_0$, called a trivial symmetry. If a symmetry generator $\mathbf{X} = \mathbf{X}_0 + \epsilon \mathbf{X}_1$ exists with $\mathbf{X}_0 \neq 0$ and $\mathbf{X}_1 \neq \mathbf{X}_0$, then it will be called non-trivial symmetry generator [32].

3 Approximate Symmetries of the Schwarzschild Spacetime

In this section, starting from the Minkowski spacetime which is maximally symmetric with 10 Killing vectors, we shall discuss approximate symmetries of the Schwarzschild spacetime. The isometries of the Minkowski spacetime form Poincare algebra, $so(1, 3) \oplus_s R^4$, where the semi-direct sum \oplus_s indicates that the sub-algebras do not commute. Also, the algebra $so(1, 3)$ is isomorphic to $so(3) \oplus_s so(3) \oplus_s R^4$, one of $so(3)$ gives conservation of angular momentum, another $so(3)$ provides conservation of spin angular momentum, R^4 gives conservation of energy and linear momentum. The symmetry generators of the Minkowski spacetime are given [33] in Appendix A. The algebra calculated from the geodesic equations for the Minkowski spacetime contains some symmetries which do not correspond to any conservation law. These symmetries arise due to re-mixing of the geodetic parameter with the Noether symmetries.

The Schwarzschild spacetime is given by

$$ds^2 = e^{\nu(r)} dt^2 - e^{-\nu(r)} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad e^{\nu(r)} = 1 - \frac{2M}{r}. \quad (12)$$

This has four isometries $\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ which provide conservation laws of energy and angular momentum. However, the conservation laws of linear and spin angular momentum are lost due to the presence of gravitational mass. The symmetry algebra of the Schwarzschild spacetime (calculated through the geodesic equations) consists of the four isometries and the dilation algebra d_2 (algebra corresponding to $\frac{\partial}{\partial s}$ and $s\frac{\partial}{\partial s}$), i.e., $so(3) \oplus R \oplus d_2$ (\oplus denotes the direct sum). In the limit of small mass of point gravitating source, $\epsilon = 2M$ [20], and using approximate symmetry analysis, all lost conservation laws are recovered as first-order trivial approximate symmetries yielding approximate conservation laws of these quantities. Also, for the more restricted equation of motion, i.e., the orbital equation, along with the symmetries

$$\mathbf{Y}_0 = u \frac{\partial}{\partial u}, \quad \mathbf{Y}_1 = \cos \phi \frac{\partial}{\partial u}, \quad \mathbf{Y}_2 = \sin \phi \frac{\partial}{\partial u}, \quad (13)$$

$$\mathbf{Y}_3 = \frac{\partial}{\partial \phi}, \quad \mathbf{Y}_4 = \cos 2\phi \frac{\partial}{\partial \phi} - u \sin 2\phi \frac{\partial}{\partial u}, \quad (14)$$

$$\mathbf{Y}_5 = \sin 2\phi \frac{\partial}{\partial \phi} + u \cos 2\phi \frac{\partial}{\partial u}, \quad (15)$$

$$\mathbf{Y}_6 = u \cos \phi \frac{\partial}{\partial \phi} - u^2 \sin \phi \frac{\partial}{\partial u}, \quad (16)$$

$$\mathbf{Y}_7 = u \sin \phi \frac{\partial}{\partial \phi} + u^2 \cos \phi \frac{\partial}{\partial u}, \quad (17)$$

there exist two non-trivial stable approximate symmetries given by

$$\mathbf{Y}_{a1} = \sin \phi \frac{\partial}{\partial u} + \epsilon(2 \sin \phi \frac{\partial}{\partial \phi} + u \cos \phi \frac{\partial}{\partial u}), \quad (18)$$

$$\mathbf{Y}_{a2} = \cos \phi \frac{\partial}{\partial u} - \epsilon(2 \cos \phi \frac{\partial}{\partial \phi} - u \sin \phi \frac{\partial}{\partial u}). \quad (19)$$

4 Stringy Charged Black Holes

In the framework of string theory, there exist number of static spherically symmetric black hole solutions. In this regard, the first solution was given by Gibbons and Maeda [34]. Later, Garfinkle, Horowitz and Strominger developed electrically and magnetically charged solutions (known as GHS solutions) [35]. Some other kinds of solutions were also found by different people. Both string theory and GR have the same uncharged solutions. The string analogues of the RN solution are obtained by solving low energy string field equations. These string analogues of the RN solution have causal structure of the Schwarzschild spacetime. In 3+1 dimensions, there exist two analogues of the RN spacetime: one is associated with magnetic charge and another deals with the electric charge. Both of these solutions are obtained by using the solution generating techniques. We evaluate exact and approximate symmetries of the geodesic and orbital equations of motion for these GHS black hole solutions.

4.1 Symmetries and Approximate Symmetries of the Stringy Electrically Charged Black Hole

The spacetime representing the electrically charged black hole solution in string theory is given [35] by

$$ds^2 = \frac{(1 - \frac{2m}{r})}{(1 + \frac{2m \sinh^2 \alpha}{r})^2} dt^2 - \frac{1}{(1 - \frac{2m}{r})} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2). \quad (20)$$

Here m is the mass of the point gravitating source and α is the electric charge of the black hole such that $\tanh \alpha = \frac{Q}{2M}$. For $\alpha = 0$, it reduces to the Schwarzschild spacetime. For this spacetime, isometry algebra is $so(3) \oplus R$ which corresponds to conservation laws of energy and angular momentum while the conservation laws of linear and spin angular momentum are lost. To recover these lost symmetries, we use the approximate Lie symmetry method.

First, we concentrate to orbital equation of motion which is obtained by restricting the motion of the particle to an arbitrarily chosen equatorial plane. It is found [20, 21] that there is a difference between the conservation laws obtained for the full system of geodesic equations and the single orbital equation of motion. To check whether this difference also holds in the case of stringy electrically charged black hole solution, we find the orbital equation of motion as follows

$$\frac{d^2 u}{d\phi^2} + u = \frac{m}{h^2} + 3mu^2 + \frac{2m \sinh^2 \alpha}{h^2} + \frac{4um^2 \sinh^4 \alpha}{h^2}, \quad (21)$$

where h is the classical angular momentum per unit mass and $u = \frac{1}{r}$. For the second order-approximate symmetries of this equation, we define the perturbation parameter as $\epsilon = 2m$. Further, we assume $m \sinh^2 \alpha \leq \epsilon^2$ which implies that $m \sinh^2 \alpha = k\epsilon^2$, where $0 < k \leq \frac{1}{4}$ (we have taken these parameters because $\alpha = 0$ leads to the Schwarzschild spacetime and hence first-order perturbed spacetime as first-order perturbed Schwarzschild spacetime and exact as Minkowski spacetime). Introducing these perturbation parameters in Eq.(20) and retaining only the terms involving ϵ^2 , the corresponding second-order perturbed spacetime will be

$$ds^2 = \left(1 - \frac{\epsilon}{r} - \frac{4k\epsilon^2}{r}\right)dt^2 - \left(1 + \frac{\epsilon}{r} + \frac{\epsilon^2}{r^2}\right)dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

Similarly, the second-order perturbed orbital equation can be written as

$$\frac{d^2 u}{d\phi^2} + u = \epsilon\left(\frac{1}{2h^2} + \frac{3}{2}u^2\right) + \epsilon^2\left(\frac{2k}{h^2}\right). \quad (22)$$

For the exact symmetry generator, we put $\epsilon = 0$ in Eq.(22) so that the exact symmetry generators are exactly the same as given by Eqs.(13)-(17).

In the first-order approximation when we retain only the terms involving ϵ and neglect $O(\epsilon^2)$, we get the same two stable and non-trivial symmetry

generators as given by Eqs.(18) and (19) along with the symmetries given by Eqs.(13)-(17) (i.e. the approximate symmetries of the Schwarzschild space-time). For the second-order approximation, when we apply the second-order prolonged operator $\mathbf{X}^{[2]}$ to the second-order perturbed orbital equation of motion,

$$\mathbf{E}: \quad u'' + u - \frac{\epsilon}{2h^2}(1 + 3u^2h^2) - \epsilon^2\left(\frac{2k}{h^2}\right) = 0,$$

retaining only the terms involving ϵ^2 and neglecting its higher powers and substituting the values of prolongation coefficients, we obtain a set of four DEs. The simultaneous solution of these DEs shows that there exist some non-trivial symmetry generators given as

$$\mathbf{Y}_{a1} = u \frac{\partial}{\partial u} + \frac{2\epsilon k}{h^2} \frac{\partial}{\partial u}, \quad (23)$$

$$\mathbf{Y}_{a2} = \sin \phi \frac{\partial}{\partial u} + \epsilon(u \cos \phi \frac{\partial}{\partial u} + 2 \sin \phi \frac{\partial}{\partial \phi}), \quad (24)$$

$$\mathbf{Y}_{a3} = \cos \phi \frac{\partial}{\partial u} + \epsilon(u \sin \phi \frac{\partial}{\partial u} - 2 \cos \phi \frac{\partial}{\partial \phi}), \quad (25)$$

$$\mathbf{Y}_{a4} = \cos 2\phi \frac{\partial}{\partial \phi} - u \sin 2\phi \frac{\partial}{\partial u} - \frac{2k\epsilon}{h^2} \sin 2\phi \frac{\partial}{\partial u}, \quad (26)$$

$$\mathbf{Y}_{a5} = \sin 2\phi \frac{\partial}{\partial \phi} + u \cos 2\phi \frac{\partial}{\partial u} + \frac{2k\epsilon}{h^2} \cos 2\phi \frac{\partial}{\partial u}, \quad (27)$$

$$\mathbf{Y}_{a6} = u \cos \phi \frac{\partial}{\partial \phi} - u^2 \sin \phi \frac{\partial}{\partial u} - \epsilon\left(\frac{3ku}{h^2} \sin \phi \frac{\partial}{\partial u}\right), \quad (28)$$

$$\mathbf{Y}_{a7} = u \sin \phi \frac{\partial}{\partial \phi} + u^2 \cos \phi \frac{\partial}{\partial u} - \epsilon\left(\frac{3ku}{h^2} \cos \phi \frac{\partial}{\partial u}\right). \quad (29)$$

Since the re-scaling of energy of test particle was seen from the approximate symmetries of the geodesic equations [21], thus we apply this approximate symmetry analysis to the full system of geodesic equations given by

$$\ddot{t} + 2 \frac{m[1 + 2 \sinh^2 \alpha - \frac{2m \sinh^2 \alpha}{r}]}{r^2(1 - \frac{2m}{r})(1 + \frac{2m \sinh^2 \alpha}{r})} \dot{t} \dot{r} = 0, \quad (30)$$

$$\begin{aligned} \ddot{r} + (1 - \frac{2m}{r}) & \left[\frac{m}{r^2(1 + \frac{2m \sinh^2 \alpha}{r})^2} + \frac{2m \sinh^2 \alpha (1 - \frac{2m}{r})}{r^2(1 + \frac{2m \sinh^2 \alpha}{r})^3} \right] \dot{t}^2 \\ & - \frac{m}{r^2(1 - \frac{2m}{r})} \dot{r}^2 - r(1 - \frac{2m}{r})(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) = 0, \end{aligned} \quad (31)$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta \cos\theta \dot{\phi}^2 = 0, \quad (32)$$

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2 \cot\theta \dot{\theta}\dot{\phi} = 0. \quad (33)$$

Now we introduce the perturbation parameters in the above equations so that the corresponding perturbed geodesic equations become

$$\mathbf{E}_1 : \quad \ddot{t} + \frac{\epsilon}{r^2}\dot{t}\dot{r} + \frac{\epsilon^2}{r^3}(1 + 4rk)\dot{t}\dot{r} = 0, \quad (34)$$

$$\begin{aligned} \mathbf{E}_2 : \quad & \ddot{r} - r(\dot{\theta}^2 + \sin^2\theta \dot{\phi}^2) + \epsilon\left[\frac{1}{2r^2}(\dot{t}^2 - \dot{r}^2)\right. \\ & \left. + (\dot{\theta}^2 + \sin^2\theta \dot{\phi}^2)\right] - \frac{\epsilon^2}{2r^3}[(1 - 4rk)\dot{t}^2 + \dot{r}^2] = 0, \end{aligned} \quad (35)$$

$$\mathbf{E}_3 : \quad \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta \cos\theta \dot{\phi}^2 = 0, \quad (36)$$

$$\mathbf{E}_4 : \quad \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2 \cot\theta \dot{\theta}\dot{\phi} = 0. \quad (37)$$

Applying the second-order prolonged operator defined by

$$\begin{aligned} \mathbf{X}^{[2]} = & \xi \frac{\partial}{\partial s} + \eta^0 \frac{\partial}{\partial t} + \eta^1 \frac{\partial}{\partial r} + \eta^2 \frac{\partial}{\partial \theta} + \eta^3 \frac{\partial}{\partial \phi} + \eta_{,s}^0 \frac{\partial}{\partial \dot{t}} + \eta_{,s}^1 \frac{\partial}{\partial \dot{r}} \\ & + \eta_{,s}^2 \frac{\partial}{\partial \dot{\theta}} + \eta_{,s}^3 \frac{\partial}{\partial \dot{\phi}} + \eta_{,ss}^0 \frac{\partial}{\partial \ddot{t}} + \eta_{,ss}^1 \frac{\partial}{\partial \ddot{r}} + \eta_{,ss}^2 \frac{\partial}{\partial \ddot{\theta}} + \eta_{,ss}^3 \frac{\partial}{\partial \ddot{\phi}} \end{aligned} \quad (38)$$

to the perturbed geodesic equations, retaining only the terms involving ϵ^2 and substituting the values of the prolongation coefficients, we obtain a system of 60 DEs. In construction of the system of determining equations for the second-order approximation, we use Eqs.(A1)-(A3) as the four exact symmetry generators and remaining six as the first-order approximate part of the symmetry generators. In this set of determining equations, out of the four constants corresponding to exact symmetry generators, two do not appear and the remaining two cancel out. However, the six constants corresponding to the first-order approximate part of the symmetry generator are present. To make this system of DEs homogenous, we have to eliminate these 6 constants. The solution obtained by back and forth substitution shows that all these constants vanish and therefore the resulting system being similar to that of the Minkowski spacetime, provides 12 second-order approximate symmetry generators. Among these generators, four are again exact used earlier

and they simply add into them while the remaining six replace the lost first-order approximate symmetries. Thus the full system has Poincare algebra $so(1, 3) \oplus_s R^4$ and hence there are no non-trivial second-order symmetries.

The exact symmetry generators consist of some symmetries associated to dilation algebra, $\xi(s) = c_0 s + c_1$. The terms involving $\xi_s(s) = c_0$ cancel out in the set of equations for the first-order approximate symmetries. However, these terms do not cancel out automatically in the second-order approximation but collect a scaling factor of $(1 + 4rk)$ for cancellation. We know that energy conservation comes from the time translational invariance and ξ is the coefficient of $\frac{\partial}{\partial s}$ (s is the proper time) in the point transformation given by Eq.(6). Thus the coefficient of c_0 corresponds to energy re-scaling factor given by

$$(1 + 4rk) = 1 + \frac{r \sinh^2 \alpha}{m}. \quad (39)$$

Here we have used the value for k .

4.2 Symmetries and Approximate Symmetries of the Stringy Magnetically Charged Black Hole

The spacetime representing the magnetically charged black hole solution is [35]

$$ds^2 = \frac{(1 - \frac{2M}{r})}{(1 - \frac{Q^2}{Mr})} dt^2 - \frac{1}{(1 - \frac{2M}{r})(1 - \frac{Q^2}{Mr})} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (40)$$

where M is the mass of the point gravitating source and Q is the magnetic charge of the black hole. For $Q = 0$, the spacetime reduces to the Schwarzschild solution. Here isometry algebra is $so(3) \oplus R$ providing the conservation laws for energy and angular momentum only. We discuss the exact and approximate symmetries of this solution by applying the same procedure as given in the previous section.

We discuss the symmetry structure of this spacetime for the orbital equation of motion given by

$$\frac{d^2 u}{d\phi^2} + u = -\frac{Q^2}{Mh^2} \left(1 - \frac{Q^2 u}{M}\right) + \frac{M}{h^2} - \frac{2Q^2 u}{h^2} + \frac{Q^2}{2Mh^2} + 3Mu^2 - 4Q^2 u^3 + \frac{3}{2M} Q^2 u^2. \quad (41)$$

For the second order-approximate symmetries of this equation, we define the perturbation parameters as $\epsilon = 2M$, $Q^2/M = k\epsilon^2$, where we have assumed $Q^2/M \leq \epsilon^2$ and therefore k is given as $0 < k \leq \frac{1}{4}$ (again these perturbation parameters are taken in order to reduce the perturbed spacetime for exact and first-order approximation to Minkowski and first-order perturbed Schwarzschild spacetimes respectively). Under these perturbations, the second-order perturbed orbital equation is

$$\frac{d^2u}{d\phi^2} + u = \epsilon\left(\frac{1}{2h^2} + \frac{3u^2}{2}\right) - \epsilon^2\left(\frac{k}{2h^2} - \frac{3ku^2}{2}\right). \quad (42)$$

For the exact symmetry generator, we take $\epsilon = 0$ in this equation and the exact symmetries are given by Eqs.(13)-(17).

If we take only the terms involving ϵ and neglect the terms of ϵ^2 and its higher powers, i.e., the first-order approximation, we get approximate symmetries of the Schwarzschild spacetime including two non-trivial stable approximate symmetry generators given by Eqs.(18) and (19). The second order approximation yields no non-trivial symmetry generator. Only the first-order symmetry generators are recovered. Hence there is no new approximate conservation law but only the previous conservation laws have been recovered.

The set of geodesic equations for this spacetime are

$$\ddot{t} + \frac{[\frac{2M}{r^2} - \frac{Q^2}{Mr^2}]}{(1 - \frac{2M}{r})(1 - \frac{Q^2}{Mr})} \dot{t}\dot{r} = 0, \quad (43)$$

$$\begin{aligned} \ddot{r} + \frac{[\frac{2M}{r^2} - \frac{Q^2}{Mr^2}](1 - \frac{2M}{r})}{2(1 - \frac{Q^2}{Mr})} \dot{t}^2 + \frac{[\frac{2M}{r^2} - \frac{4Q^2}{r^3} + \frac{Q^2}{Mr^2}]}{2(1 - \frac{2M}{r})(1 - \frac{Q^2}{Mr})} \dot{r}^2 \\ - r(1 - \frac{2M}{r})(1 - \frac{Q^2}{Mr})[\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2] = 0, \end{aligned} \quad (44)$$

$$\ddot{\theta} + \frac{2}{r} \dot{r}\dot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 = 0, \quad (45)$$

$$\ddot{\phi} + \frac{2}{r} \dot{r}\dot{\phi} + 2 \cot \theta \dot{\theta}\dot{\phi} = 0. \quad (46)$$

Introduce the perturbation parameters defined earlier, the corresponding second-order perturbed geodesic equations become

$$\ddot{t} + \frac{\epsilon}{r^2} \dot{t}\dot{r} + \frac{\epsilon^2}{r^3} (1 - kr) \dot{t}\dot{r} = 0, \quad (47)$$

$$\ddot{r} - r(\dot{\theta} + \sin^2 \theta \dot{\phi}^2) + \epsilon \left[\frac{1}{2r^2} (\dot{t}^2 - \dot{r}^2) + \dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 \right] - \frac{\epsilon^2}{2r^3} [(1 + rk)(\dot{t}^2 + \dot{r}^2) - 2kr^3(\dot{\theta} + \sin^2 \theta \dot{\phi}^2)] = 0, \quad (48)$$

while the last two equations remain the same under these perturbations. Here we use the same procedure as in the previous section and obtain the same symmetries as for the electrically charged black hole but in this case the re-scaling factor is different. The energy re-scaling factor turns out to be $(1 - rk) = 1 - r \frac{Q^2}{M^3}$.

5 Summary and Discussion

In this paper, we have discussed the approximate symmetries of the stringy charged black hole solutions. These solutions have the isometry algebra $so(3) \oplus R$ while the system of geodesic equations have $so(3) \oplus R \oplus d_2$. Firstly, we have found the second-order approximate symmetries of the orbital equation of motion. For the stringy magnetically charged solution, there does not exist any non-trivial approximate symmetry generator. Only exact and the first-order symmetry generators are recovered as the second-order trivial symmetry generators. For the electrically charged solution, the exact and first-order symmetry generators are found to be the same as for the Schwarzschild spacetime. However, for the second-order approximation, there exist some non-trivial symmetry generators given by Eqs.(23)-(29).

Secondly, we have calculated symmetries of the geodesic equations for both the spacetimes. For these charged solutions, the exact and first-order approximate part of the symmetry generator turn out to be the same as that for the Schwarzschild spacetime. For the second-order approximation to symmetry generators, we get no non-trivial symmetries. We have only recovered the lost conservation laws as the second-order approximate conservation laws. But unlike the Schwarzschild spacetime, from the perturbed geodesic equations, we have obtained some re-scaling factor. The re-scaling factor for the electrically charged solution is $(1 + 4rk) = 1 + \frac{r \sinh^2 \alpha}{m}$.

Using Møller's prescription, the energy distribution is given by [36]

$$\frac{mr^2}{(r + 2m \sinh^2 \alpha)^2} \left[1 + 2 \left(1 - \frac{m}{r} \right) \sinh^2 \alpha \right].$$

The re-scaling of force for the electrically charged solution calculated through the pseudo-Newtonian formalism [37] is

$$-\frac{\frac{m}{r^2}(1 + \sinh^2 \alpha - \frac{2m}{r} \sinh^2 \alpha)}{(1 - \frac{2m}{r})(1 + \frac{2m}{r} \sinh^2 \alpha)}$$

which reduces to the Schwarzschild solution for charge to be zero. Also, the re-scaling factor of energy vanishes for α (charge) to be zero (i.e. the Schwarzschild solution). It is mentioned here that all the three expressions are r dependent.

For the magnetically charged solution, the re-scaling factor turns out to be $(1 - rk) = 1 - r\frac{Q^2}{M^3}$. The re-scaling of force by using the pseudo-Newtonian becomes

$$-\frac{\frac{M}{r^2}(1 - \frac{Q^2}{2M^2})}{2r(1 - \frac{2M}{r})(1 - \frac{Q^2}{Mr})}.$$

The expression for energy calculated through Einstein's prescription is [38]

$$E(r) = M - \frac{1}{2}r - \frac{M}{2Q^2}r^2 - \frac{M^2}{2Q^4}r^3 - \frac{M^3}{2Q^6}r^4 + O(r^5).$$

In the limit of $Q = 0$, it must reduce to energy for the Schwarzschild space-time but in the above expression, this limit leads to $E(r) \rightarrow \infty$. However, in the limiting case, the re-scaling factor leads to the result of the Schwarzschild spacetime. Further, energy calculated through Møller's prescription is given by [36]

$$E(r) = \frac{(2M^2 - Q^2)r}{2(Mr - Q^2)}$$

which reduces to energy of the Schwarzschild spacetime for $Q = 0$. We would like to mention here that both the re-scaling factors (obtained in our case) are r dependent. However, it turns out to be more simple evaluated by using the approximate symmetry as compared to expressions evaluated through other approaches. Also, for the electrically charged solution, it is seen that there exist some non-trivial symmetries found through orbital equation of motion. This fact is in contrast to the RN spacetime where no non-trivial part of symmetry generator exists but the re-scaling factor is independent of r .

In literature [20, 21], a difference between the conservation laws for full system of geodesic equations and single orbital equation of motion is noted.

We conclude that this difference also holds for the stringy charged solutions. It is found that when some symmetries are lost at one order (exact or the first-order approximation) then they are recovered at the next order (at least to the second-order) as trivial approximate symmetries.

It would be worthwhile to investigate the approximate symmetry generators and the concept of energy re-scaling for the rotating stringy black hole solutions which contain electric, magnetic and both charges [39, 40]. For the complete understanding of this procedure, it would be interesting to examine the colliding gravitational waves [41]. Work on these lines is in progress.

Appendix A

The symmetry generators of the Minkowski spacetime are given as follows:

$$\mathbf{X}_0 = \frac{\partial}{\partial t}, \quad \mathbf{X}_1 = \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi}, \quad (\text{A1})$$

$$\mathbf{X}_2 = \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi}, \quad (\text{A2})$$

$$\mathbf{X}_3 = \frac{\partial}{\partial \phi}, \quad (\text{A3})$$

$$\mathbf{X}_4 = \sin \theta \cos \phi \frac{\partial}{\partial r} + \frac{\cos \theta \cos \phi}{r} \frac{\partial}{\partial \theta} - \frac{\csc \theta \sin \phi}{r} \frac{\partial}{\partial \phi}, \quad (\text{A4})$$

$$\mathbf{X}_5 = \sin \theta \sin \phi \frac{\partial}{\partial r} + \frac{\cos \theta \sin \phi}{r} \frac{\partial}{\partial \theta} + \frac{\csc \theta \cos \phi}{r} \frac{\partial}{\partial \phi}, \quad (\text{A5})$$

$$\mathbf{X}_6 = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}, \quad (\text{A6})$$

$$\begin{aligned} \mathbf{X}_7 = & r \sin \theta \cos \phi \frac{\partial}{\partial t} + t \left(\sin \theta \cos \phi \frac{\partial}{\partial r} + \frac{\cos \theta \cos \phi}{r} \frac{\partial}{\partial \theta} \right. \\ & \left. - \frac{\csc \theta \sin \phi}{r} \frac{\partial}{\partial \phi} \right), \end{aligned} \quad (\text{A7})$$

$$\begin{aligned} \mathbf{X}_8 = & r \sin \theta \sin \phi \frac{\partial}{\partial t} + t \left(\sin \theta \sin \phi \frac{\partial}{\partial r} + \frac{\cos \theta \sin \phi}{r} \frac{\partial}{\partial \theta} \right. \\ & \left. + \frac{\csc \theta \cos \phi}{r} \frac{\partial}{\partial \phi} \right), \end{aligned} \quad (\text{A8})$$

$$\mathbf{X}_9 = r \cos \theta \frac{\partial}{\partial t} + t \left(\cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right). \quad (\text{A9})$$

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